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Development of the Tangent Modulus from the Euler-Problem to Nonsmooth Materials

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Abstract. History and development of the tangent modulus from the origins to the recent nonsmooth damaging versions are presented. Load history and stability analyses of structures of nonlinear reversible or irreversible materials are based on the concept of tangent modulus. Generally, instantaneously changing tangent modulus is needed and the solution yields iteration process. In the case of inelastic problems, the switch from loading to unloading of the material behaviour results in nonsmooth material functions. Nonsmooth, generally saw-tooth like behaviour happens in composite, laminated or rock type materials, or in the interaction of concrete and the reinforcement, too. Recently, damage and localization are in the focus of structural analyses, extending the tangent modulus to the negative cases, as well. Consequently, an overview of the history and development of the tangent modulus containing the recent modifications seems to be necessary. On the other hand, the more than a century long history of the tangent modulus is a marvellous study of the parallel development of mechanics and mathematics, by following the mutual inspiring effect of them through the activity of such pioneers like P.D. Panagiotopoulos in creating Nonsmooth Mechanics.

Key words: Damaging joints or materials; Elastic; Locking; Nonconvex problems; Nonsmooth materials; Plastic; Tangent modulus

1. Introduction

The classical elastic load history and stability analyses are based on the convex and smooth elastic potential. The nearly also classical elastic-plastic analyses need also smooth and convex potential by reducing the problem to a quasi-elastic analysis. The modern inelastic stability analyses including strain softening and damage are extended to nonconvex potentials by the generalization of the tangent modulus. However, the condition of smoothness is further on required by using the concept of linear comparison solid.

The nonsmooth characteristics of strain energy functionals can be derived from two facts. On one hand, it can be caused by the change of material phases: the switch from loading to unloading is an original nonsmooth characteristics of any inelastic behaviour. Even to avoid nonsmoothness, the concept of linear comparison solid has been introduced by Hill (1958). On the other hand, the nonsmooth properties of strain energy can be resulted by nonsmooth functions of material laws directly, due to original or approximate polygon like characteristics.

By using the tools of nonsmooth analysis, we can get over the difficulties of nonsmoothness of both type. By the help of the *nonsmooth tangent modulus*, global load history or stability analyses of nonlinearly elastic or inelastic structures can be investigated.

In this paper the development and modifications of the tangent modulus is presented. Analytical and numerical aspects and application of the nonsmooth versions are considered.

2. The history of the tangent modulus

Since the tangent modulus plays a key role in structural analyses, it seems to be necessary to overview the development of it. Thus we can see how the tangent modulus changed during about one and a half century, from a simple material constant to an indicator tensor of dissipative systems. In the history of the tangent modulus we follow the state of art given by Bruhns in (1984).

2.1. THE BIRTH OF THE TANGENT MODULUS

The concept of tangent modulus is resulted by the development of plasticity, namely, the plastic bifurcation problems. Plastic column buckling, the possible bifurcation of the structure was in the focus of interest in the past, however, the progress in the subject was not smooth.

The problem to calculate the critical value of the load on the top of the column, when the straight configuration becomes unstable, was first solved by Euler in 1744, by assuming linear elastic material. However if the stress in the column exceeds the yield limit, plastic flow will occur, and the original elastic modulus is not valid any more.

A typical stress-strain curve $\sigma = \sigma(\epsilon)$ can be seen in Figure 1a where the slope

$$E_t = d\sigma/d\epsilon \tag{1}$$

of the curve beyond the elastic limit σ_0 is the tangent modulus. The tangent modulus is still the function of strains, since $E_t(\epsilon) = d\sigma(\epsilon)/d\epsilon$, thus for the stress-strain function, a bilinear idealization seen in Figure 1b is often used. This model seems to be advantageous since $E_t = const$, however, the nonsmoothness in $\sigma(\epsilon)$, namely, the jump in the slope passing through the yield limit, causes difficulties.

The first revised formula for the critical load was suggested by Engesser in 1889 by using the tangent modulus E_t in the formula of Euler. A bit later, in 1891 by Considére and in 1894 by Jasinski, an important observation was made by pointing out that in a buckling mode, in one part of the section a purely elastic strain reversal will occur, while in the remaining part, plastic loading will continue.

Considére introduced the so-called reduced modulus E_r for which $E \geqslant E_r \geqslant E_t$. Some examples were performed later by Kármán in 1910.

Then, for a long period, the reduced modulus theory was accepted in the subject, until in 1946-47 a considerable progress has been investigated by Shanley.

Shanley (1946), by the help of a simple model demonstrated the important distinction between uniqueness and stability. He recognised that the tangent modulus load is the lowest possible bifurcation load. At this load, the straight configuration loses its uniqueness but not its stability. Moreover, Shanley has written the tangent modulus in the form in which it is used in our time, too, as

$$\delta\sigma = \begin{cases} E\,\delta\epsilon & \text{for elastic loading or elastic–plastic unloading} \\ E_t\,\delta\epsilon & \text{for plastic loading} \end{cases} \tag{2}$$

By an other important observation of Shanley that at the instant of bifurcation there is no change in the load, Shanley quasi made an advance of the grounds of the concept of linear comparison solid of Hill. However, it took another decade until the continuum theory of bifurcation was laid down by the fundamental paper of Hill in 1958.

2.2. EXTENSION OF THE TANGENT MODULUS TO THE CONTINUUM THEORY

Until the famous paper of Hill in 1958, the tangent modulus was considered as the property of a material point only. Hill was who extended the concept of the tangent modulus to the whole body or structure by characterising the ‘resistance’ of the body by the tangent modulus (1962, 1967, 1978).

By following Bruhns (1984), here we refer only to those results of Hill which are in closed connection with the tangent modulus. Hill suggested for the rate constitutive relations of bodies with elastic-plastic material and finite deformations as follows

$$\dot{\sigma} = E\dot{\epsilon} - \frac{\alpha}{h}(\lambda\dot{\epsilon})\lambda = \dot{\sigma}(\dot{\epsilon}), \tag{3}$$

in the case of smooth yield surface and associated plastic deformations. Here $\dot{\sigma}$ is the tensor of the so-called objective stress increments and E is the tensor of the instantaneous elastic moduli while λ represents the normals to the yield hypersurface interfaces separating the domains of elastic and plastic behaviour. Here α is a positive function of hardening as follows

$$\alpha = \begin{cases} 1 & \text{if } \lambda\dot{\epsilon} \geqslant 0 \\ 0 & \text{if } \lambda\dot{\epsilon} < 0 \end{cases} \tag{4}$$

as an indicator of the regimes of plastic loading and unloading. When the stress lies within the yield surface, the material is purely elastic thus $\alpha = 0$.

On the basis of Hill's tangent modulus, a strain rate potential

$$W(\dot{\epsilon}) = \frac{1}{2}\dot{\epsilon}E\dot{\epsilon} - \frac{1}{2}\frac{\alpha}{h}(\lambda\dot{\epsilon})^2 = W_e(\dot{\epsilon}) + W_p(\dot{\epsilon}, \alpha) \tag{5}$$

can be introduced as the potential function of the stress rates

$$\dot{\sigma} = \frac{\partial W(\dot{\epsilon})}{\partial\dot{\epsilon}} = E\dot{\epsilon} - \frac{\alpha}{h}\lambda\dot{\epsilon}\lambda. \tag{6}$$

However, this function is nonsmooth with respect to the strain rates. It has continuous first derivative and partly continuous second derivative. The jump in the second derivative is due to the jump in the indicator α. Without using the tools of nonsmooth analysis the nonsmooth potential function (5) cannot be handled.

Based on the observation of Shanley according to which in column buckling at the instant of bifurcation unloading is absent, Hill introduces a special material called *linear comparison solid* of the nonlinearly inelastic material with the property that unloading is excluded through α=1. Thus, he could avoid the nonsmoothness by obtaining a smooth potential

$$W(\dot{\epsilon}) = \frac{1}{2}\dot{\epsilon}E\dot{\epsilon} - \frac{1}{2h}(\lambda\dot{\epsilon})^2 = W_e(\dot{\epsilon}) + W_p(\dot{\epsilon}). \tag{7}$$

Hill's results have a great importance. Any solution of the rate boundary value problem for the elastic-plastic solid is unique when the uniqueness of the analogous boundary value problem for the comparison solid is assured.

Hill's results are related to elastic-plastic behaviour. However, in modern stability analyses the strain softening and damage, moreover, the strain localization became even more important. Thus, the results of Hill have recently been extended to these cases, on the basis of the thermodynamics and by using the tools of functional analysis.

2.3. THERMODYNAMIC GENERALIZATION OF THE TANGENT MODULUS

The thermodynamic extension of the tangent modulus is the merit of Nguyen (1990, 1993), Halphen and Nguyen (1975) by introducing the *generalized time-independent standard dissipative material*. This concept is the basis of the modern bifurcation theories. General constitutive relations of strain softening materials are given by the authors Rice (1971, 1976), Rice and Rudnicki (1980), Raniecki and Bruhns (1981). Modern mathematical description is given by Benallal, Billardon and Geymonat (1989), Billardon and Doghri (1989), Benallal, Billardon and Geymonat (1993).

Assuming small isothermal strains according to Benallal, Billardon and Geymonat (1989, 1993), the behaviour of *generalized time-independent standard dissipative materials* can be characterised by three thermodynamical potential functions, the *free energy* $\Psi = \Psi(\epsilon, \alpha, T)$, the *reversibility function* $f(A, \alpha, T)$ and the *dissipation function* $F(A, \dot{\alpha}, T)$, all expressed in term of the strains ϵ, the internal kinematical variables α, and the temperature T.

Generally the *free energy*

$$\Psi = \Psi(\epsilon, \alpha, T) \tag{8}$$

is the potential function of the statical type thermodynamic state variables, the stresses σ, the thermodynamic forces A, and the entropy s as follows

$$\sigma = \rho\frac{\partial\Psi}{\partial\epsilon}, \qquad A = -\rho\frac{\partial\Psi}{\partial\alpha}, \qquad s = -\rho\frac{\partial\Psi}{\partial T}. \tag{9}$$

The *reversibility* is governed by the *function* $f(A, \alpha, T)$ specifying the *domain of reversibility*

$$C(\alpha) = \{A | f(A, \alpha, T) \leqslant 0\}. \tag{10}$$

The *dissipation function* $F(A, \dot{\alpha}, T)$ yields the normality law

$$\dot{\alpha} = \lambda\frac{\partial F}{\partial A} = N_c(a), \tag{11}$$

where $N_c(a)$ is the outer normal vector of the convex set $C(\alpha)$ at A. The nonnegative multiplier $\lambda \geqslant 0$ results from the *consistency condition* $\dot{f} = 0$:

$$\lambda =< \frac{\frac{\partial f}{\partial A} \circ \Lambda : \dot{\epsilon}}{h} >, \tag{12}$$

where

$$h = \frac{\partial f}{\partial A} \circ \Pi \circ \frac{\partial F}{\partial A} - \frac{\partial F}{\partial \alpha} \circ \frac{\partial F}{\partial A} > 0, \tag{13}$$

moreover,

$$\Lambda = -\rho\frac{\partial^2\Psi}{\partial\alpha\partial\epsilon}, \qquad \Pi = \rho\frac{\partial^2\Psi}{\partial\alpha\partial\alpha}. \tag{14}$$

Here the symbol $\circ$ denotes the scalar product while $\otimes$ denotes the tensorial product between tensors. Symbol : denotes the double tensor contraction, and $< x >=$ max $< x, 0 >$ ensures the nonnegativity.

Considering *isothermal* process, the *time-independent standard dissipative material* can be characterised by three simplified potential functions $\Psi(\epsilon, \alpha)$, $f(A, \alpha)$ and $F(A, \dot{\alpha})$. In this case the temperature and the entropy can be eliminated, thus, the thermodynamic state laws (9) yield

$$\sigma = \rho\frac{\partial\Psi}{\partial\epsilon}, \qquad A = -\rho\frac{\partial\Psi}{\partial\alpha}. \tag{15}$$

By the help of these functions the general thermodynamic form of the tangent modulus can be obtained. Taking the time derivative $\dot{\sigma} = \partial\sigma/\partial t$ as quasi-static velocity into account, the rate constitutive relation can be written in the form

$$\dot{\sigma} = \mathbf{L}(\epsilon, \alpha) : \dot{\epsilon}, \tag{16}$$

where the operator **L** is the tangent modulus as follows

$$\mathbf{L} = \begin{cases} \mathbf{E} & \text{if } f(A,\alpha) = 0 \text{ and } \mathbf{b} : \mathbf{E} : \dot{\epsilon} < 0 \\ \mathbf{E} - \frac{(\mathbf{E}:\mathbf{a})\otimes(\mathbf{b}:\mathbf{E})}{h} & \text{if } f(A,\alpha) = 0 \text{ and } \mathbf{b} : \mathbf{E} : \dot{\epsilon} \geqslant 0' \end{cases} \tag{17}$$

in which

$$\mathbf{E} = \rho\frac{\partial^2\Psi}{\partial\epsilon\partial\epsilon} \tag{18}$$

is the tensor of elastic moduli, and the tensors

$$\mathbf{a} = \mathbf{E}^{-1} : \Lambda^T \circ \frac{\partial F}{\partial A} \quad \text{and} \quad \mathbf{b} = \frac{\partial f}{\partial A} \circ \Lambda : \mathbf{E}^{-1} \tag{19}$$

are related to the law of normality and the domain of reversibility.

The general form (17) of the tangent modulus contains equally any nonlinear inelastic and even strain softening or damaging characteristics of materials. Let us consider now the most important special cases.

In the case of *elastic-plastic materials*, function Ψ is the Helmholtz free energy. According to Benallal, Billardon and Geymonat (1989), for elastic-plastic materials *with non-associated flow law*, f is the yield function and F is the plastic potential function, consequently, tensors **a** and **b** are the gradients of the plastic potential and the yield surface, respectively. Thus, the tangent modulus is modified to

$$\mathbf{L} = \begin{cases} \mathbf{E} & \text{if } f(A,\alpha) = 0 \text{ and } \mathbf{b} : \mathbf{E} : \dot{\epsilon} < 0 \\ \mathbf{E} - \frac{(\mathbf{E}:\mathbf{a})\otimes(\mathbf{b}:\mathbf{E})}{H+\mathbf{a}:\mathbf{E}:\mathbf{b}} & \text{if } f(A,\alpha) = 0 \text{ and } \mathbf{b} : \mathbf{E} : \dot{\epsilon} \geqslant 0 \end{cases} \tag{20}$$

where H is the generalized strain-hardening modulus being positive, zero or negative for strain hardening, perfect or strain softening plasticity, respectively, see Neilsen and Schreyer (1993). For *associated flow law*, even $f = F$, that is **a**=**b**, consequently, the tangent modulus (17) is simplified to

$$\mathbf{L} = \begin{cases} \mathbf{E} & \text{if } f(A,\alpha) = 0 \text{ and } \mathbf{a} : \mathbf{E} : \dot{\epsilon} < 0 \\ \mathbf{E} - \frac{(\mathbf{E}:\mathbf{a})\otimes(\mathbf{a}:\mathbf{E})}{H+\mathbf{a}:\mathbf{E}:\mathbf{a}} & \text{if } f(A,\alpha) = 0 \text{ and } \mathbf{a} : \mathbf{E} : \dot{\epsilon} \geqslant 0 \end{cases} \tag{21}$$

Moreover, in the case of *linearly elastic and perfectly plastic materials*, **a**=**b**=**1** and $\partial f/\partial\alpha = 0$, consequently, the tangent modulus yields

$$\mathbf{L} = \begin{cases} \mathbf{E} & \text{for elastic loading, elastic–plastic unloading} \\ \mathbf{0} & \text{for plastic loading} \end{cases} \tag{22}$$

In the case of *damaging materials*, a scalar damage coefficient D is introduced among the internal variables. According to, for *elastic-damaging materials*, let D be $0 \leqslant D \leqslant D_{cr} \leqslant +\infty$, fulfilling the basic condition that for $D = 0$, the material is perfectly elastic, and for $D = D_{cr}$, the material is perfectly damaged. The stiffness of the material is characterised by a function $g(D)E$ where E is the initial elastic modulus. Thus, by means of these functions a rate constitutive law of the damaging material can be obtained detailed in Benallal, Billardon and Geymonat (1993). For a very simple model of the *elastic-damaging material*, Benallal, Billardon and Geymonat (1989) suggest $0 \leqslant D \leqslant 1$ and the free energy to be $\rho\Psi(\epsilon, D) = \frac{1}{2}(1 - D)\epsilon : E : \epsilon$, moreover, $k(D) = Q_o + MD$ where Q_o and M are material constants. Thus they suggest the tangent modulus to be as

$$\mathbf{L} = (1 - D)\mathbf{E} - \frac{(\mathbf{E} : \epsilon) \otimes (\epsilon : \mathbf{E})}{M} \tag{23}$$

for the elastic-damaging material.

Other tangent moduli are obtained for continuum damage material on the basis of fracture mechanics by Janson and Hult (1977) and Del Piero and Sampaio (1989).

2.4. EXTENSION OF THE TANGENT MODULUS TO LOCALIZATION

Recently, the phenomenon of localization, namely if the damage or strong strain softening is localized into a small, quasi a zero volume of the body, is in the focus of the research. As we have seen, by considering elasto-plastic material with non-associated flow law, the tangent modulus is modified to (20), where the part

$$\mathbf{D}^{ep} = \mathbf{E} - \frac{(\mathbf{E} : \mathbf{a}) \otimes (\mathbf{b} : \mathbf{E})}{H + \mathbf{a} : \mathbf{E} : \mathbf{b}} \tag{24}$$

is the elasto-plastic tangent modulus tensor. If the plastification leads to strain softening localized into a small part of the body, the tangent modulus needs certain modification.

Based on the results of the analysis of the shear band localization (see the references in the state of art by Neilsen and Schreyer (1993), and by Szabó (1998)), the modified tangent modulus yields the second order *acoustic or localization tensor* specified by

$$\mathbf{Q} = \mathbf{n} \cdot \mathbf{D}^{ep} \cdot \mathbf{n}, \tag{25}$$

where $\mathbf{n}$ is the normal vector of the surface of localized deformations or discontinuity, separating the zone of localized deformations from the rest of the body. The acoustic tensor $\mathbf{Q}$ and the normal vector $\mathbf{n}$ is used as the basis of classification of the bifurcation modes, that is for distinguishing the diffuse (nonlocalized) and discontinuous (localized) bifurcation forms.

3. The nonsmooth tangent modulus

After the short review of the tangent modulus, we focus ourselves to the nonsmooth characteristics of it. Before detailing the consequences of the nonsmooth material behaviour, the short history of the nonsmooth potential theory is considered.

3.1. DEVELOPMENT OF THE NONSMOOTH POTENTIAL THEORY

For conservative mechanical systems the stability conclusions can be drawn simply from the properties of the total potential energy functional, by the *Lagrange–Dirichlet* theorem Thompson and Hunt (1973, 1984). The fundamental stability statements are based on the classical potential law

$$\sigma(\epsilon) = \frac{\partial W(\epsilon)}{\partial \epsilon} \tag{26}$$

where functional $W(\epsilon)$ is the smooth and convex strain energy density, $\sigma = \{\sigma_{ij}\}$ and $\epsilon = \{\epsilon_{ij}\}$ are the stress and strain tensors, respectively.

If the functional $W(\epsilon)$ is nonsmooth but the material is reversible, the classical potential law (26) can be extended to polygonal elastic cases. Panagiotopoulos pointed out in (1988 p. 85) that while the 'Smooth Mechanics' is based on the notion of the classical potential, the 'Nonsmooth Mechanics' is concerned with the nonsmooth and/or nonfinite convex or nonconvex superpotentials.

The generalization of the classical potential law to nonsmooth but convex potentials named superpotential was introduced by Moreau (1963, 1968) by using the tools of the convex analysis. The convexity of an energy function implies the monotonicity of the concerning stress-strain relation. Variational principles related to such kind of problems have the form of variational inequalities. In order to overcome the constraint of monotonicity, the notion of nonconvex superpotential was introduced by Panagiotopoulos (1981) by using the generalized gradient of Clarke (1975) and the results of Rockafellar (1970) leading to the hemivariational inequalities in mechanical applications. In his pioneer book (1985) Panagiotopoulos laid down the foundations of the 'Nonsmooth Mechanics' and established the substationarity laws of mechanics. So he obtained the generalized substationarity principles for nonconvex potentials (1985, 1988). Further generalizations of hemivariational inequalities and applications are given by Moreau and Panagiotopoulos (1988), and later Naniewicz and Panagiotopoulos (1995).

The term of nonsmoothness in sense of the definitions of Panagiotopoulos (1985) is based on the Lipschitzian property of functions. Simply saying, for a break type discontinuity of a function *f(x)* at *x*, the Lipschitzian condition at *x* fulfils, while for a jump type discontinuity it does not. The existence of both the subdifferential $\partial f(x)$ and the generalized gradient $\bar{\partial} f(x)$ requires the Lipschitzian property of the function at *x*. A point x_o is called a substationarity point of $f(x)$ if it is a solution of the multivalued equation named inclusion

$$0 \in \bar{\partial} f(x) \tag{27}$$

where the generalized gradient $\bar{\partial} f(x)$ of Clarke is a set being never empty if $f(x)$ is Lipschitzian at x. If $f(x)$ is convex then $\bar{\partial} f(x)$ coincides with the subdifferential: $\bar{\partial} f(x) = \{\text{grad } f(x)\}$, moreover if it is also continuously differentiable at x, then $\bar{\partial} f(x) = \{\text{grad } f(x)\}$.

If the material is reversible but the strain energy functional $W(\epsilon)$ is nonsmooth, the conservative stresses can be obtained from the inclusion

$$\sigma(\epsilon) \in \partial W(\epsilon) \tag{28}$$

named superpotential law, introduced by Moreau in (1968). According to Panagiotopoulos (1985) $\partial W(\epsilon)$ is the subdifferential of the superpotential $W(\epsilon)$, a multivalued mapping as the generalization of the classical potential law, see Panagiotopoulos (1981).

Further generalization given by Panagiotopoulos in 1981 aimed to extend the potential law to nonmonotone material behaviour. If the material has a nonmonotone constitutive law but the material is reversible, the nonconvex superpotential law

$$\sigma(\epsilon) \in \bar{\partial} W(\epsilon) \tag{29}$$

is valid, by using the generalized gradient of Clarke. In this way, a wide range of decreasing and even a saw-tooth form material behaviour can be dealt with.

According to Panagiotopoulos (1988), the classical linear elasticity laws can be replaced by nonlinear elasticity laws $\sigma(\epsilon) = \partial W(\epsilon)/\partial\epsilon$ or more generally, by the monotone nonsmooth law $\sigma(\epsilon) \in \partial W(\epsilon)$, and even by the nonmonotone nonsmooth law $\sigma(\epsilon) \in \bar{\partial} W(\epsilon)$, if the material is reversible.

In his basic works, Panagiotopoulos deals also with the potential law of dissipative mechanical systems in (1983, 1985, 1988). From the view of the nonsmoothness and nonconvexity on thermodynamic bases, he obtain mainly the same stationarity conclusion as mentioned above. By introducing a general nonsmooth thermodynamic potential of dissipation and applying the superpotential law of Clarke, he gets to the generalization of the hypothesis of normal dissipation. He states that in the case of dissipation or unloading, an incremental analysis has to be applied. He deals generally with strongly nonmonotone and nonsmooth cases like saw-tooth behaviour and damage, but in aspect first of all the equilibrium and not the stability.

On the basis of the results of Panagiotopoulos, we can extend the concept of tangent modulus to the nonsmooth cases, including the combination of plastic and locking behaviour, the so-called generalized conditional joints as well.

3.2. GENERALIZED CONDITIONAL JOINTS AS SUBDIFFERENTIAL CONSTITUTIVE MODELS

The concept of the nonsmooth tangent modulus can be related to the so-called *locking materials*, according to Suquet (1985). The initial stiffness of this materials during a loading process increases and finally, the material can become even perfectly rigid seen in Figure 2a. In Figure 2b the perfectly locking behaviour is illustrated. In spite of the fact that this type of materials are reversible, they can be handled similarly to the irreversible problems. The locking behaviour can be combined with the plastic behaviour seen in Figure 2c, yielding to the family of the so-called *conditional joints* described first by Kaliszky (1975). Further generalization of the conditional joints by considering them as subdifferential material property was given by Kurutz (1985, 1987), namely, by considering the locking behaviour as the dual version of the reversible plastic characteristics. Stability conclusions due to nonsmooth behaviour has been analysed also by Kurutz (1991, 1993, 1994, 1996).

In Figure 2d the saw-tooth type stress-strain diagram of some recently used materials and structures can be seen. Composite, fibre-reinforced, laminated or rock-like materials, adhesive connections, moreover, the interaction of concrete with the steel reinforcement are characterised by this type of functions.

The typical nonsmooth material behaviour seen in Figure 2c and 2d can be considered as generalized conditional joints governed by subsequent locking and plastic yield conditions, respectively, specifying the convex sets in the function space R^6

$$K_i(x_k) = \{\epsilon_{ij}(x_k) | g_i(\epsilon_{ij}(x_k)) \leqslant 0\} \quad x_k \in V, \epsilon_{ij} \in R^6 \quad i = 1, 2, \dots, m \tag{30}$$

and

$$K_j^c(x_k) = \{\sigma_{ij}(x_k) | f_j(\sigma_{ij}(x_k)) \leqslant 0\} \quad x_k \in V, \sigma_{ij} \in R^6 \quad j = 1, 2, \dots, n \tag{31}$$

represented by six-dimensional convex hypersurfaces illustrated symbolically in Figure 3. In the figure, three surfaces can be seen, representing a subsequent elastic-plastic-locking-plastic behaviour seen in Figure 2c. In (30) and (31) *i* and *j* are the number of subsequent locking and yield conditions, respectively.

To extend the constitutive law of the generalized conditional joints to R^6, the associated indicator functionals of the convex sets K and K^c are specified as

$$J_K(\epsilon_{ij}) = \begin{cases} \Phi g(\epsilon_{ij}) = 0, & \text{for } \epsilon_{ij} \in K \\ \infty & \text{for } \epsilon_{ij} \notin K \end{cases} \tag{32}$$

and

$$J_K^c(\sigma_{ij}) = \begin{cases} \Lambda f(\sigma_{ij}) = 0, & \text{for } \sigma_{ij} \in K^c \\ \infty & \text{for } \sigma_{ij} \notin K^c \end{cases} \tag{33}$$

respectively, representing the normality or orthogonality law, where $\Phi \geqslant 0$ and $\Lambda \geqslant 0$ are the multipliers of the locking stress and plastic strain increments, respectively. On the other hand, these sign-dependent variables can be considered as the Lagrange-multipliers of the sign-dependent locking and yield conditions $g(\epsilon_{ij}) \leqslant 0$ and $f(\sigma_{ij}) \leqslant 0$, respectively.

By means of the indicators, the strain and stress energy density functionals can be constructed respectively as

$$W(\epsilon_{ij}) = W_0(\epsilon_{ij}) + J_K(\epsilon_{ij}) \qquad \text{and} \qquad W^c(\sigma_{ij}) = W_0^c(\sigma_{ij}) + J_K^c(\sigma_{ij}) \tag{34}$$

Thus, the multivalued constitutive law representing substationarity yields the inclusions

$$\begin{aligned} \epsilon_{ij} \in \partial W^c(\sigma_{ij}) &\equiv \partial W_0^c(\sigma_{ij}) + \partial J_K^c(\sigma_{ij}) \\ &\equiv \partial W_0^c(\sigma_{ij}) + \begin{cases} \Lambda \partial f(\sigma_{ij}) = \Lambda f_{ij}, & \text{for } \sigma_{ij} \in K^c \\ 0 & \text{for } \sigma_{ij} \notin K^c \end{cases} \end{aligned} \tag{35}$$

$$\begin{aligned} \sigma_{ij} \in \partial W(\epsilon_{ij}) &\equiv \partial W_0\ (\epsilon_{ij}) + \partial J_K(\epsilon_{ij}) \\ &\equiv \partial W_0\ (\epsilon_{ij}) + \begin{cases} \Phi \partial g(\epsilon_{ij}) = \Phi g_{ij}, & \text{for } \epsilon_{ij} \in K \\ 0 & \text{for } \epsilon_{ij} \notin K \end{cases} \end{aligned} \tag{36}$$

for plastification and locking process, where f_{ij} and g_{ij} are the gradients of the yield and locking convex hypersurfaces, respectively.

The modified variational problem extended to the sign-dependent variables can symbolically be illustrated by means of the Hu-Washizu principle seen in Figure 4. as a surface to be extremized under the inequality subsidiary conditions $g(\epsilon_{ij}) \leqslant 0$ and $f(\sigma_{ij}) \leqslant 0$. This leads to a constrained extremum problem, where the domain of the possible solutions is restricted by the inequality side conditions. Consequently, the stationarity condition leads to variational inequalities. The numerical solution of the load history analysis can be solved as mathematical programming problem.

3.3. ONE-DIMENSIONAL NONSMOOTH TANGENT MODULI

By applying the concept of tangentially equivalent elastic structure of Bazant and Cedolin (1991, p. 635), the responses of an inelastic problem can be solved in small loading steps by a series of quasielastic analysis, taking the inelastic constitutive law as a thermodynamic equation of state into account. Thus, for a small step $d\epsilon$, the increment of the strain energy can be considered elastic, so the strain energy $W(\epsilon)$ at ϵ can as the potential function of the stresses $\sigma(\epsilon)$ be considered, that is, the classical potential law can be applied. This principle can be extended to the nonsmooth cases, to the superpotential law, as well.

By extending the tangent modulus to the unloading, too, the following stress-strain function including the case $d\epsilon = 0$ is considered

$$\sigma(\epsilon) = \begin{cases} \sigma_0(\epsilon) = k_0(\epsilon)(\epsilon - \epsilon_0(\epsilon)) & \text{if } d\epsilon \leqslant 0 \\ \sigma_t(\epsilon) = k_t(\epsilon)(\epsilon - \epsilon_t(\epsilon)) & \text{if } d\epsilon \geqslant 0 \end{cases} \tag{37}$$

in which the function $\sigma_0(\epsilon)$ concerns both the elastic-plastic or elastic-plastic-damage unloading, while the function $\sigma_t(\epsilon)$ belongs to loading only, seen in Figure 5a. Note that function $\sigma_t(\epsilon)$ represents Hill's linear comparison solid of the original nonlinear material. Here $k_t(\epsilon)$ and $k_0(\epsilon)$ are the loading and unloading moduli related to the linear functions $\sigma_t(\epsilon)$ and $\sigma_0(\epsilon)$ at ϵ, respectively. Strain values $\epsilon_t(\epsilon)$ and $\epsilon_0(\epsilon)$ are the intersections of the straight lines $\sigma_t(\epsilon)$ and $\sigma_0(\epsilon)$ with the axis ϵ, respectively. As we can see in Figure 5a, all these values are continuously changing in term of ϵ, but at a fixed strain value ϵ, they are constant, that is

$$\sigma(\epsilon) = \begin{cases} k_0(\epsilon - \epsilon_0) & \text{if } d\epsilon \leqslant 0 \\ k_t(\epsilon - \epsilon_t) & \text{if } d\epsilon \geqslant 0 \end{cases} \tag{38}$$

Consequently, for obtaining the tangent modulus at any fixed ϵ, for the nonsmooth relation (38), subdifferentiation is applied

$$\bar{K}_t(\epsilon) \equiv \bar{\partial}\sigma(\epsilon) = \begin{cases} k_0 & \text{if } d\epsilon < 0 \\ [k_t, k_0] & \text{if } d\epsilon = 0 \\ k_t & \text{if } d\epsilon > 0 \end{cases} \tag{39}$$

yielding the nonsmooth tangent modulus seen in Figure 2b, as a multivalued function by forming an interval of $[k_t, k_0]$ at the condition $d\epsilon = 0$. Here k_0 is the initial elastic modulus and $k_t = k_t(\epsilon)$ is the actual tangent of function $\sigma(\epsilon)$ Thus, the actual occurring stiffness $K_t(\epsilon)$ is the element of the set $\bar{K}_t(\epsilon)$ that is $K_t(\epsilon) \in \bar{K}_t(\epsilon)$.

However, the actual tangent $k_t = k_t(\epsilon)$ is changing with changing ϵ which makes the solution difficult. That is why, in certain problems, polygonal approximation seems to be reasonable. Moreover, some materials show originally polygonal characteristics. In this case, the solution gives directly the correct results, of course. Figure 6a shows a polygonal material function. Here we consider break type functions without jumps.

Let each segment i of the polygonal material law in Figure 6a be specified by the relating modulus k_t^i as the constant slope of segment i, and by the strain constant ϵ_t^i as the intersection of segment i and the axis ϵ. Thus, at the break point $\epsilon = \epsilon_i$, the material function $\sigma(\epsilon)$ can be written in the form

$$\sigma(\epsilon)^i = \begin{cases} k_t^{i-1}(\epsilon - \epsilon_t^{i-1}) & \text{if } d\epsilon \leqslant 0 \\ k_t^i(\epsilon - \epsilon_t^i) & \text{if } d\epsilon \geqslant 0 \end{cases} \tag{40}$$

which is similar to the function (38) since the segment preceding the break point $\epsilon = \epsilon_i$ can as unloading path be considered. Practically, during a loading process, case $d\epsilon < 0$ belongs to unloading only. If the material is reversible, the unloading is represented in (40) by the modulus k_t^{i-1}, while for irreversible materials $k_t^{i-1} = 0$ for $d\epsilon < 0$.

The nonsmooth tangent modulus $\bar{K}_t(\epsilon)^i$ for $\epsilon = \epsilon_i$, seen in Figure 6b can be obtained by subdifferentiating the function $\sigma(\epsilon)^i$ at $\epsilon = \epsilon_i$, relating to both loading and unloading

$$\bar{K}_t(\epsilon)^i \equiv \bar{\partial}(\sigma(\epsilon)^i) = \begin{cases} k_t^{i-1} & \text{if } d\epsilon < 0 \\ [k_t^i, k_t^{i-1}] & \text{if } d\epsilon = 0 \\ k_t^i & \text{if } d\epsilon > 0 \end{cases} \tag{41}$$

The concept of the nonsmooth tangent modulus of polygonal material behaviour can be extended to the *strain softening*, namely, to *damage* problems, too. As a typical damage property, in the case of active damage loading, the loading moduli k_t^i are negative. In contrast to the elastic-plastic unloading, in damaging cases, the unloading and reloading moduli are given individually. Consider a polygonal function of an elastic-plastic-damaging material seen in Figure 7. Also in the case of damaging materials, the unloading paths are linear, but in contrast to the plastic unloading, unloading occurs with different elastic moduli. Thus, the unloading moduli k_t^u are changing depending on the actual strains.

Consider now the lump like material nonsmoothness seen in Figure 2d and in Figure 8. Jump like material characteristics can occur in both strain softening or strain hardening phases, like in composite materials or locking behaviour, respectively. Figure 8 represents the behaviour of the perfectly rigid-plastic material. In this case, the unloading and reloading take place in a perfectly rigid manner under the condition $d\epsilon = 0$, manifested in a jump. The material behaviour is characterized by the inclusion

$$\sigma(\epsilon) \in \bar{\sigma}(\epsilon) = \begin{cases} \sigma_2 & \text{if } d\epsilon < 0 \\ [\sigma_2, \sigma_1] & \text{if } d\epsilon = 0 \\ \sigma_1 & \text{if } d\epsilon > 0 \end{cases} \tag{42}$$

according which, independently of ϵ, the actual stresses are the elements of the set of stresses related equally to loading, unloading and reloading.

For obtaining the tangent modulus, this function needs to be subdifferentiated. However, since this function has jumps at any $d\epsilon = 0$, the Lipschitz condition does not fulfil, so nor the subdifferential of Moreau, nor the generalized gradient of Clarke exists. Still, if we want to obtain the tangent modulus in such kind of Heaviside type material functions, a distributional derivative is applied. Thus the generalized nonsmooth tangent modulus yields

$$\bar{K}_t(\epsilon) \equiv \bar{\partial}(\bar{\sigma}(\epsilon)) = \begin{cases} 0 & \text{if } d\epsilon \neq 0 \\ \pm(\sigma_1 - \sigma_2)\delta(d\epsilon) & \text{if } d\epsilon = 0 \end{cases} \tag{43}$$

where $\delta(d\epsilon)$ is the Dirac impulse, see Keener (1988). For the condition $d\epsilon = 0$, the tangent modulus forms an interval of indefinite length, namely, in the case of loading (unloading) it tends to the positive (negative) infinite.

Naturally, an arbitrary jump $[\sigma_{i-1}, \sigma_i]$ in the material function can equally happen for a reversible or an irreversible material at any strain value. Consider now a locking material with a jump σ_{i-1}, σ_i, both preceded and followed by elastic behaviour seen in Figure 2c. The nonsmooth tangent modulus at ϵ then reads

$$\bar{K}_t(\epsilon) \equiv \bar{\partial}(\bar{\sigma}(\epsilon)) = \begin{cases} k_t^{i-1} & \text{if } d\epsilon < 0 \\ \pm(\sigma_i - \sigma_{i-1})\delta(d\epsilon) & \text{if } d\epsilon = 0 \\ k_t^i & \text{if } d\epsilon > 0 \end{cases} \tag{44}$$

since the unloading paths are equal to the loading ones.

Construct now the nonsmooth tangent modulus related to a discrete structural model.

4. The nonsmooth structural tangent modulus

Let us consider isothermal deformations of a time-independent solid body subject to a quasi-static conservative loading program. Any material property is assumed to vary smoothly in the geometric space, while the material function in itself is nonsmooth in the function space.

By supposing that the body in the initial configuration occupies a spatial domain Ω_0 and is bounded by the smooth surface Γ_0, let us consider that in the volume Ω_0 the body forces F_i, and on a nonzero part Γ_{p0} of the surface Γ_0 the surface tractions P_i, while on the complementary part Γ_{u0}, the displacements u_i are specified. Let us assume a scalar loading parameter λ to be varied continuously and infinitely slowly in time.

Let us consider Lagrangian description where S_{ij} is the second Piola–Kirchhoff stress tensor and E_{ij} is the Lagrange–Green strain tensor.

Generally, any nonlinear structural analyses are based on the *principle of incremental virtual work* representing equilibrium condition of a given state of the load history analysis. The equilibrium condition can be extended to *nonsmooth materials* as an inclusion

$$\begin{aligned} 0 \in \delta\Delta\bar{L} \;=\; & \int_{\Omega_0} (\bar{S}_{ij} + d\bar{S}_{ij})\,\delta\Delta E_{ij}\,d\Omega_0 - \int_{\Omega_0} (\lambda F_{i0} + d\lambda F_{i0})\,\delta\Delta u_i\,d\Omega_0 \\ & - \int_{\Gamma_{p0}} (\lambda P_{i0} + d\lambda P_{i0})\,\delta\Delta u_i\,d\Gamma_0, \end{aligned} \tag{45}$$

where the term Δ represents the total, *d* the first order increments and δ the variation. The nonlinear and nonsmooth material is specified by the nonsmooth function $\bar{S}_{ij}(E_{mn})$ obtained by the inclusion of the superpotential law

$$\bar{S}_{ij}(E_{mn}) \in \bar{\partial}_{ij} W(E_{mn}), \tag{46}$$

while the increments of the nonsmooth stresses are specified by

$$d\bar{S}_{ij}(E_{mn}) \in \bar{\partial}_{kl}\bar{S}_{ij}(E_{mn})dE_{kl} \equiv \bar{\partial}_{kl}(\bar{\partial}_{ij}W(E_{mn}))dE_{kl} \equiv \bar{K}_{ijkl}(E_{mn})dE_{kl}, \tag{47}$$

where $W(E_{mn})$ is the nonsmooth nonconvex superpotential, $\bar{K}_{ijkl}(E_{mn})$ is the nonsmooth multivalued material tangent modulus tensor.

The Lagrange–Green strain tensor is *smooth* in term of the displacement gradients $u_{i,j}$

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \tag{48}$$

in which linear and nonlinear parts can be distinguished. In the case of large displacement gradients, that is $u_{i,j} >> 0$, large or finite strains, while in the case of small displacement gradients, if $u_{i,j} << 1$, small or infinitesimal strains are distinguished.

In (45) we need the total increment of large strains related to the *n*-th equilibrium configuration as

$$\begin{aligned}\Delta E_{ij} &= dE_{ij} + d^2E_{ij} = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + u^n_{k,i}\,\Delta u_{k,j} \\ &\quad + u^n_{k,j}\,\Delta u_{k,i} + \Delta u_{k,i}\Delta u_{k,j},\end{aligned} \tag{49}$$

containing the fist and second order increments of the large strains at u_i^n.

In (45) the first variation of the total increment of the large strains is needed, too, as

$$\begin{aligned}\delta\Delta E_{ij} &= \delta dE_{ij} + \delta d^2E_{ij} = \frac{1}{2}(\delta\Delta u_{i,j} + \delta\Delta u_{j,i} + u^n_{k,i}\,\delta\Delta u_{k,j} \\ &\quad + u^n_{k,j}\,\delta\Delta u_{k,i} + \Delta u_{k,i}\,\delta\Delta u_{k,j} + \delta\Delta u_{k,i}\,\Delta u_{k,j})\end{aligned} \tag{50}$$

where the variation of the first and second increments can be distinguished. The increments and variations of the displacement gradients $u_{i,j}$ can be analyzed after the discretization only.

The displacement function u_i *for a single finite element within the body* can be expressed in term of both geometric and functional coordinates **X** and **q**, respectively, as

$$\underset{(3)}{\mathbf{u}} = \mathbf{u}(\underset{(3)}{\mathbf{X}}, \mathbf{q}) = \begin{bmatrix} u_1(\mathbf{X},\mathbf{q}) \\ u_2(\mathbf{X},\mathbf{q}) \\ u_3(\mathbf{X},\mathbf{q}) \end{bmatrix} = \begin{bmatrix} u_1(X_1, X_2, X_3; q_1, q_2, \dots, q_r) \\ u_2(X_1, X_2, X_3; q_1, q_2, \dots, q_r) \\ u_3(X_1, X_2, X_3; q_1, q_2, \dots, q_r) \end{bmatrix} \tag{51}$$

where $\mathbf{X}$ are the (local) coordinates of the discretized *geometric space* (the body in itself), and $\mathbf{q}$ are the coordinates of the discretized *function space*, while r is the number of generalized coordinates (finite degree of freedom) of the elements.

Here we distinguish small/large displacements, functions $\mathbf{u}$ to be linear/nonlinear in $\mathbf{q}$, respectively. Practically, in the case of large displacements, parameters $\mathbf{q}$ contain rotational elements, that is, trigonometrical relations in $\mathbf{u}$, in term of $\mathbf{q}$. For small displacements, functions $\mathbf{u}$ are linear in $\mathbf{q}$, thus, the variables $\mathbf{X}$ and $\mathbf{q}$ in (51) can be separated by the linear combination

$$u_i = \sum_{k=1}^{m} q_i^k N_i^k(\mathbf{X}), \tag{52}$$

where $N_i^k(\mathbf{X})$ are the interpolation or shape functions corresponding to the nodal points of number m within the element. Expression (52) leads to the classical basic expression of the linear finite element displacement method

$$\mathbf{u}(\underset{(3)}{\mathbf{X}}, \mathbf{q}) = \underset{(3,r)}{\mathbf{N}(\mathbf{X})}\ \underset{(r)}{\mathbf{q}}, \tag{53}$$

where matrix $\mathbf{N}(\mathbf{X})$ contains the shape functions $N_i^k(\mathbf{X})$ of the classical linear FEM approach.

In the case of nonlinear displacements, when the direct separation (52) cannot be applied, incrementally linear analysis is needed, that is, the linear combination (53) can be applied to the increments of the n-th configuration only. By considering the increments of the large displacements as $\Delta\mathbf{u} = d\mathbf{u} + d^2\mathbf{u}$, for the incremental version of (53) we have

$$d\underset{(3)}{\mathbf{u}} = \left.\frac{\partial \mathbf{u}(\mathbf{X}, \mathbf{q})}{\partial q_j}\right|_n dq_j = \underset{(3,r)}{\mathbf{H}_n}\, d\underset{(r)}{\mathbf{q}}, \tag{54}$$

moreover, the nonlinearity of the displacements is represented by

$$d^2\underset{(3)}{\mathbf{u}} = \left.\frac{1}{2}\frac{\partial^2 \mathbf{u}(\mathbf{X}, \mathbf{q})}{\partial q_j \partial q_k}\right|_n dq_j\, dq_k = \frac{1}{2} d\underset{(r)}{\mathbf{q}^T}\ \underset{(r,3,r)}{\mathbf{W}_n}\ d\underset{(r)}{\mathbf{q}}, \tag{55}$$

which are the first and second order increments of the large displacements, respectively, related to the n-th configuration. Matrix $\mathbf{H}_n$ has $3\times r$ elements, while matrix $\mathbf{W}_n$ is three dimensional of measure $r\times 3\times r$. At a certain load level the incrementally linear relation (54) can as the basic relation of the nonlinear finite element displacement method be considered, while matrix $\mathbf{W}_n$ represents the nonlinear geometry.

The variation of the increments of large displacements are as follows

$$\delta\Delta\mathbf{u} = \mathbf{H}_n\, \delta d\mathbf{q} + d\mathbf{q}^T \mathbf{W}_n\, \delta d\mathbf{q} \tag{56}$$

where the variation of the first and second increments can be distinguished as $\delta d\mathbf{u} = \mathbf{H}_n\,\delta d\mathbf{q}$ and $\delta d^2\mathbf{u} = d\mathbf{q}^T\mathbf{W}_n\,\delta d\mathbf{q}$. After discretizing the displacements, the matrix version of the Green-Lagrange strains can be obtained in the form

$$\underset{(6)}{\mathbf{E}} = \underset{(6)}{\mathbf{E}(\mathbf{u})} = \underset{(6,3)}{\mathbf{A}}\,\underset{(3)}{\mathbf{u}} + \frac{1}{2}\,\underset{(3)}{\mathbf{u}^T}\,\underset{(3,6,3)}{\mathbf{G}}\,\underset{(3)}{\mathbf{u}} \tag{57}$$

where $\mathbf{E}$ are in vector arrangement as $\mathbf{E}^T = [E_{11}\ E_{22}\ E_{33}\ 2E_{12}\ 2E_{13}\ 2E_{23}]$, moreover, $\mathbf{A}$ and $\mathbf{G}$ are differential operators of the geometric space $\mathbf{X}$, concerning the displacement gradients represented by the linear term $\mathbf{Au}$ in the small (infinitesimal) strains, and, by the nonlinear term $1/2\ \mathbf{u}^T\ \mathbf{G}\,\mathbf{u}$ in the case of large (finite) strains. Matrix $\mathbf{G}$ is three-dimensional, consisting of six layers of sub-matrices of measure 6×3.

By considering the discrete versions of the increments and variations of large strains in term of large displacements, we can obtain

$$\Delta\mathbf{E} = (\mathbf{AH}_n + \mathbf{u}_n^T\mathbf{GH}_n)\,d\mathbf{q} + 1/2\,d\mathbf{q}^T\,(\mathbf{AW}_n + \mathbf{u}_n^T\mathbf{GW}_n + \mathbf{H}_n^T\mathbf{GH}_n)d\mathbf{q}, \tag{58}$$

in which the first and second increments can be distinguished as $d\mathbf{E} = (\mathbf{AH}_n + \mathbf{u}_n^T\mathbf{GH}_n)d\mathbf{q} + 1/2d\mathbf{q}^T\,(\mathbf{AW}_n + \mathbf{u}_n^T\mathbf{GW}_n)d\mathbf{q}$ and $d^2\mathbf{E} \cong 1/2d\mathbf{q}^T\mathbf{H}_n^T\mathbf{GH}_n d\mathbf{q}$ where the latter is obtained by eliminating the higher than second order terms in d$\mathbf{q}$. Moreover, the variation of increments are

$$\delta\Delta\mathbf{E} = (\mathbf{AH}_n + \mathbf{u}_n^T\mathbf{GH}_n)\,\delta d\mathbf{q} + d\mathbf{q}^T\,(\mathbf{AW}_n + \mathbf{u}_n^T\mathbf{GW}_n + \mathbf{H}_n^T\mathbf{GH}_n)\,\delta d\mathbf{q}, \tag{59}$$

in which $\delta d\mathbf{E} = (\mathbf{AH}_n + \mathbf{u}_n^T\mathbf{GH}_n)\,\delta d\mathbf{q} + d\mathbf{q}^T\,(\mathbf{AW}_n + \mathbf{u}_n^T\mathbf{GW}_n)\delta d\mathbf{q}$ and $\delta d^2\mathbf{E} \cong d\mathbf{q}^T\mathbf{H}_n^T\mathbf{GH}_n\,\delta d\mathbf{q}$. These results are valid to nonlinear strains and nonlinear displacements. By applying linear strains with nonlinear displacements, or, conversely, nonlinear strains with linear displacements, moreover both linear strains and displacements, the above increments and variations can be simplified. More details can be seen in Kurutz (1999).

By considering the discrete versions of the state variable functions, the incremental virtual work (45) related to nonsmooth materials, for a single element of the assemblage can be obtained in matrix form as

$$\begin{aligned}\delta\Delta\bar{L}^e \ \equiv\ & \int_{\Omega_0^e}(\bar{\mathbf{S}}^T + \Delta\mathbf{E}^T\bar{\mathbf{D}}_t)\,\delta\Delta\mathbf{E}\,d\Omega_0 - \int_{\Omega_0^e}(\lambda\mathbf{F}_0^T + d\lambda\mathbf{F}_0^T)\delta\Delta\mathbf{u}\,d\Omega_0 \\ & - \int_{\Gamma^e}(\lambda\mathbf{P}_0^T + d\lambda\mathbf{P}_0^T)\,\delta\Delta\mathbf{u}\,d\Gamma_0\end{aligned} \tag{60}$$

where $\bar{\mathbf{D}}_t$ contains the nonsmooth material tangent moduli. Due to the nonlinearity of the strains and displacements, the expression (60) is fully nonlinear, thus, further concepts of linearization are necessary, see Kurutz (1999). The linearized and homogenized form of the incremental virtual work in the case of nonsmooth material with large strains and large displacements yields a set

$$\begin{aligned}\delta\Delta\bar{L}^e \equiv\; & d\mathbf{q}^T \left\{ \int_{\Omega_0} \mathbf{H}_n^T (\mathbf{A}^T + \mathbf{G}^T \mathbf{B}\mathbf{u}_n)\, \bar{\mathbf{D}}_\mathbf{t}\, (\mathbf{u}_n^T \mathbf{G} + \mathbf{A})\mathbf{H}_n \, d\Omega_0 \right. \\ & + \int_{\Omega_0} \bar{\mathbf{S}}^{-T} (\mathbf{A}\mathbf{W}_n + \mathbf{u}_n^T \mathbf{G}\mathbf{W}_n + \mathbf{H}_n^T \mathbf{G}\mathbf{H}_n)\, d\Omega_0 \\ & \left. - \int_{\Omega_0} \lambda \mathbf{F}_0^T \mathbf{W}_n \, d\Omega_0 - \int_{\Gamma_0} \lambda \mathbf{P}_0^T \mathbf{W}_n \, d\Gamma_0 \right\} \delta d\mathbf{q} \\ & - d\lambda \left\{ \int_{\Omega_0^e} \mathbf{F}_0^T \mathbf{H}_n \, d\Omega_0 + \int_{\Gamma_{p0}} \mathbf{P}_0^T \mathbf{N}_n \, d\Gamma_0 \right\} \delta d\mathbf{q} \end{aligned} \tag{61}$$

related to a single element of the discretized body, where

$$\begin{aligned}\bar{\mathbf{k}}_t^n \equiv\; & \int_{\Omega_0^e} \mathbf{H}_n^T (\mathbf{A}^T + \mathbf{G}^T \mathbf{u}_n)\, \bar{\mathbf{D}}_t^n\, (\mathbf{u}_n^T \mathbf{G} + \mathbf{A})\mathbf{H}_n \, d\Omega_0 \\ & + \int_{\Omega_0^e} \bar{\mathbf{S}}_n^T (\mathbf{A}\mathbf{W}_n + \mathbf{u}_n^T \mathbf{G}\mathbf{W}_n + \mathbf{H}_n^T \mathbf{G}\mathbf{H}_n)\, d\Omega_0 \\ & - \int_{\Omega_0^e} \lambda \mathbf{F}_0^T \mathbf{W}_n \, d\Omega_0 - \int_{\Gamma_0^e} \lambda \mathbf{P}_0^T \mathbf{W}_n \, d\Gamma_0 \end{aligned} \tag{62}$$

is the *multivalued elementary tangent stiffness matrix* of nonsmooth nonlinear case, related to the n-th equilibrium configuration. By extending the principle of incremental virtual work to the total element assemblage (after some coordinate transformation and other operations not detailed here), the incremental finite element equilibrium relation of the entire structure can be obtained as an inclusion

$$0 \in d\mathbf{q}^T \bar{\mathbf{K}}_t^n \, \delta d\mathbf{q} - d\mathbf{R}_n^T \, \delta d\mathbf{q} = (d\mathbf{q}^T \bar{\mathbf{K}}_t^n - d\mathbf{R}_n^T)\; \delta d\mathbf{q}, \tag{63}$$

where $\bar{K}_n^T$ is the *nonsmooth structural tangent stiffness matrix* which can basically be divided into three parts. The first part is the so-called *nonsmooth material tangent stiffness*

$$\bar{\mathbf{k}}_{\text{tang}}^{\text{mat}} = \int_{\Omega_0^e} \mathbf{H}_n^T (\mathbf{A}^T + \mathbf{G}^T \mathbf{u}_n)\, \bar{\mathbf{D}}_t^n\, (\mathbf{u}_n^T \mathbf{G} + \mathbf{A})\mathbf{H}_n \, d\Omega_0 \tag{64}$$

containing the nonsmooth material tangent moduli including unloading, moreover, linear and nonlinear strains. The second part, named *nonsmooth geometric stiffness matrix*

$$\bar{\mathbf{k}}_{\text{geom}}^{\text{stress}} = \int_{\Omega_0^e} \bar{\mathbf{S}}_n^T (\mathbf{A}\mathbf{W}_n + \mathbf{u}_n^T \mathbf{G}\mathbf{W}_n + \mathbf{H}_n^T \mathbf{G}\mathbf{H}_n)\, d\Omega_0 \tag{65}$$

represents the actual nonsmooth stresses and the geometric nonlinearities including both nonlinear strains and displacements. The third part, named *loading stiffness matrix*

$$\mathbf{k}_{\mathrm{geom}}^{\mathrm{load}} = \int_{\Omega_0^e} \lambda \mathbf{F}_0^T \mathbf{W}_n \, d\Omega_0 + \int_{\Gamma_0^e} \lambda \mathbf{P}_0^T \mathbf{W}_n \, d\Gamma_0 \tag{66}$$

is associated with displacement nonlinearity, being smooth function.

The incremental analysis is based on the *tangent stiffness matrix*. By using the detailed forms of the discrete strains and displacements, different forms of the tangent stiffness matrix can be obtained. In the following tablet the main versions of the nonsmooth tangent stiffness matrix modified by the different linearization and approximation concepts are summarized.

Nonsmooth Elementary Tangent Stiffness		
Nonsmooth material	Large strains	Small strains
Large displacements	$\int_{\Omega_0^e} \mathbf{H}_n^T(\mathbf{A}^T + \mathbf{G}^T\mathbf{u}_n)\bar{\mathbf{D}}_t^n(\mathbf{u}_n^T\mathbf{G} + \mathbf{A})\mathbf{H}_n d\Omega_0$ $+ \int_{\Omega_0^e} \bar{\mathbf{S}}_n^T(\mathbf{A}\mathbf{W}_n + \mathbf{u}_n^T\mathbf{G}\mathbf{W}_n + \mathbf{H}_n^T\mathbf{G}\mathbf{H}_n)d\Omega_0$ $- \int_{\Omega_0^e} \lambda\mathbf{F}_0^T\mathbf{W}_n d\Omega_0 - \int_{\Gamma_0^e} \lambda\mathbf{P}_0^T\mathbf{W}_n d\Gamma_0$	$\int_{\Omega_0^e} \mathbf{H}_n^T\mathbf{A}^T\bar{\mathbf{D}}_t^n\mathbf{A}\mathbf{H}_n d\Omega_0$ $+ \int_{\Omega_0^e} \bar{\mathbf{S}}_n^T\mathbf{A}\mathbf{W}_n d\Omega_0$ $- \int_{\Omega_0^e} \lambda\mathbf{F}_0^T\mathbf{W}_n d\Omega_0$ $- \int_{\Gamma_0} \lambda\mathbf{P}_0^T\mathbf{W}_n d\Gamma_0$
Small displacements	$\int_{\Omega_0^e} \mathbf{N}^T(\mathbf{A}^T + \mathbf{G}^T\mathbf{u}_n)\bar{\mathbf{D}}_t^n(\mathbf{u}_n^T\mathbf{G} + \mathbf{A})\mathbf{N} d\Omega_0$ $+ \int_{\Omega_0^e} \bar{\mathbf{S}}_n^T(\mathbf{N}^T\mathbf{G}\mathbf{N})d\Omega_0$	$\int_{\Omega_0^e} \mathbf{N}^T\mathbf{A}^T\bar{\mathbf{D}}_t^n\mathbf{A}\mathbf{N} d\Omega_0$

5. Application of the nonsmooth tangent modulus

Figure 9a shows the classical structural model of stable-symmetric bifurcation problen, a rigid bar supported by a rotational spring. The moment M in the spring represents the stress, while the rotation ϑ represents the strain variable. Here the spring is supposed to be nonlinear and nonsmooth, specified by the polygonal material function $\bar{M}(\vartheta)$ seen in Figure 9b. According to Kurutz (1993, 1994, 1996) a nonlinear function of an elastic material can be approximated by a polygonal composed by the segments

$$M(q)_j = c_j(q - \vartheta_j) \quad 0 < c_j < \infty \; j = 1, 2, \ldots, n \tag{67}$$

related to the *j*-th segment of the material polygonal. Here c_j is the slope of the *j*-th segment, and ϑ_j is the intersection of the *j*-th segment and the strain axis, seen in Figure 9b.

The functional finitization needs to introduce the generalized coordinates. In these simple case of one degree of kinematical freedom, the vector $\mathbf{q}$ has a single element $q = \vartheta$, the rotation at the support hinge.

Figure 9a shows the applied one parameter vertical load $F = \lambda F_0$ where $F_0 = 1$. Due to the single vertical load, the displacement function is represented by the vertical displacement of the top of the cantilever, which, by assuming perfect nonlinear displacements, yields $u = l(1 - \cos q)$. The nonlinear strain function is $\vartheta(u) = \arccos(1 - u/l)$ while $\vartheta(u(q)) = \arccos(1 - u/l) = q$.

In this way, by considering nonlinear nonsmooth damaging material

$$M(q)_j = c_j(q - \vartheta_j) 0 < c_j < \infty \quad j = 1, 2, \ldots, n \tag{68}$$

represented by the polygonal seen in Figure 10a as the lower envelope of the functions in (68), due to the damaging characteristics. The nonsmooth equilibrium path forms the lower envelope of the component functions again, that is

$$\bar{\lambda}(q) = \min \left\{ \frac{1}{F_0 l} \left\{ \begin{array}{ll} \frac{c_j(q-\vartheta_j)}{\sin q} & \text{for } 0 < c_j < \infty \\ \left[\frac{M_{j,j-1}}{\sin \vartheta_j}, \frac{M_{j,j+1}}{\sin \vartheta_j} \right] & \text{for } c_j = \infty \qquad j = 1, 2, \ldots, n \\ \frac{M_j}{\sin q} & \text{for } c_j = 0 \end{array} \right. \right\} \tag{69}$$

illustrated in Figure 10b for the right hand side deflections $0 \leqslant q \leqslant \pi$. As it is concluded in Kurutz (1993, 1994, 1996), for softening material phases the lower envelope, for hardening material phases the upper envelope yields the equilibrium path.

For qualifying the stability of the equilibrium paths, we need the second subdifferential of the nonsmooth superpotential. The nonsmooth function of the structural tangent stiffness $\bar{K}(q)$ are seen in Figure 10c. For differentiable points the tangent stiffness consists of a single value, while for subdifferentiable points it forms an interval. The jump like function of the associated *multivalued tangent stiffness matrix* is as follows:

$$\bar{K}(q) = \left\{ \begin{array}{ll} c_j \left(1 - \frac{q-\vartheta_j}{\mathrm{tg} q}\right) & \text{for } 0 < c_j < \infty \\ \left[c_j \left(1 - \frac{q-\vartheta_j}{\mathrm{tg}\vartheta_{j,j+1}}\right), c_{j+1} \left(1 - \frac{q-\vartheta_{j+1}}{\mathrm{tg}\vartheta_{j,j+1}}\right) \right] & \text{for } 0 < c_j < \infty \\ & \text{and } q = \vartheta_{j+1} \\ \text{positive intervals} & \text{for } c_j = \infty \\ \frac{M_j}{\mathrm{tg} q} & \text{for } c_j = 0 \end{array} \right. \tag{70}$$

containing intervals associated with the break points of the material polygonal.

In the one dimensional case, the stability of equilibrium at any state represented by the points of the equilibrium paths seen in Figure 10b, can be qualified by a simple sign control of the related functions of the tangent stiffness. In order to find the critical state and critical load, we consider the inclusion

$$0 \in \det \bar{K}(q) \tag{71}$$

knowing that the determinant of an interval matrix forms an interval, too. In this one-dimensional case, this matrix has a single element which is equal to its determinant in itself. Thus, the critical states can be seen in Figure 10b, as well.

Figure 11a shows a model where the phenomenon of unloading can also be demonstrated. The structure has the total length l consisting of two rigid elements of length αl and $(1-\alpha)l$ with the ratio $\xi = \alpha/(1-\alpha)$ specifying the position of the middle joint. This system has also one degree of kinematical freedom. Let the parameter q be the angle of rotation $q = \vartheta_A$ of the support joint A.

Let the elastic-plastic-damaging behaviour of the joints be represented by the material function $M(\vartheta)$ seen in Figure 11b, prescribed uniformly for the three springs. In contrast to the plastic unloading, in the damage zone the unloading moduli can be specified individually.

The displacement function $\mathbf{u}(q)$ and the strain function $\epsilon(\mathbf{u})$ are nonlinear, since

$$\mathbf{u}(q) = \begin{bmatrix} u_{Bx} \\ u_{By} \\ u_{Cy} \end{bmatrix} = \begin{bmatrix} l\alpha \sin q \\ l\alpha(1-\cos q) \\ l\alpha(1-\cos q) + l(1-\alpha)(1-\cos(\arcsin(\xi \sin q))) \end{bmatrix} = \begin{bmatrix} u_1(q) \\ u_2(q) \\ u_3(q) \end{bmatrix} \tag{72}$$

and

$$\epsilon(\mathbf{u}) = \begin{bmatrix} \vartheta_A \\ \vartheta_B \\ \vartheta_C \end{bmatrix} = \begin{bmatrix} \arcsin(u_1/\alpha l) \\ \arcsin(u_1/\alpha l) + \arcsin(u_1/((1-\alpha)l)) \\ \arcsin(u_1/((1-\alpha)l)) \end{bmatrix} \tag{73}$$

moreover

$$\epsilon(\mathbf{u}(\mathbf{q})) = \begin{bmatrix} \vartheta_A \\ \vartheta_B \\ \vartheta_C \end{bmatrix} = \begin{bmatrix} q \\ q + \arcsin(\xi \sin q) \\ \arcsin(\xi \sin q) \end{bmatrix} \tag{74}$$

Let us consider the case of $\alpha = 1/3$, that is $\xi = 0.5$, namely, if the hinge B is in the lower third of the total height l. In Figure 12a the structural material behaviour is illustrated, as the resultant of the three springs, reduced to the support hinge A, due to the choice $q = \vartheta_A$.

Figure 12a shows the nonsmooth function of the structural material behaviour resulted by the simultaneously different material phases of each joints A, B and D. The simultaneity of the different material phases depends on the actual strains at the joints controlled by the actual rotations $\epsilon(q)$, namely, the compatibility transformations (74). As a consequence of the softening behaviour, the structural moment function forms the lower envelope of the component functions in Figure 12a.

The concerning nonsmooth equilibrium paths $\bar{\lambda}(q)$ of the structure are seen in Figure 12b. For the sake of simplicity this time we follow the behaviour of the structure in the right-hand side interval $0 \leqslant q \leqslant \pi$ again. We can observe that certain paths can not be realized since there is no such coincidence of the material phases of the three springs. The equilibrium path $\bar{\lambda}(q)$ forms the lower envelope of the *realized* component functions in Figure 12b. At $\vartheta_A = \pi/2$ of the joint A, joint C starts to be unloaded causing certain energy release.

In Figure 12c the multivalued function of the structural tangent stiffness is illustrated. Due to the gradual strain softening, the tangent stiffness tends to be negative indicating instabilities.

6. Conclusions

Nonsmooth material and structural tangent moduli are in the focus of this paper. After a historical review, where a century long development of the tangent modulus was detailed, we introduced the nonsmooth version of it, based on the pioneer work of P.D. Panagiotopoulos.

As a conclusion, the tangent modulus containing both material loading and unloading is always multivalued. The nonsmooth tangent modulus related to the break points of a material polygon yields intervals of finite length, while related to the jumps of a material function leads intervals of infinite length concerning to the Dirac-impulse. The nonsmooth tangent modulus of polygonal material behaviour can be applied to the cases of strain softening and damage too.

The concept of nonsmooth material tangent modulus can be extended to the whole structure, leading to nonsmooth structural tangent modulus. The generalized nonsmooth structural tangent modulus is multivalued. In the case of uniaxial material behaviour, it forms a diagonal interval matrix. For a break (jump) type material discontinuity, the intervals are finite (infinite).

One-dimensional illustrations for simple discrete structures with uniaxial material laws helped to prove the advantage of the nonsmooth material and structural tangent modulus.

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